Semistability of Nonlinear Systems Having a Continuum of Equilibria and Time-Varying Delays

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Abstract

In this paper, we develop a semistability analysis framework for nonlinear systems with time-varying delays with applications to stability analysis of multiagent dynamic networks with consensus protocols in the presence of unknown heterogeneous time-varying delays along the communication links. We show that for such a nonlinear system having a continuum of equilibria, if the system asymptotically converges to a constant time-delay system and this new system is semistable, then the original time-varying delay system is semistable, provided that the delays are just bounded, not necessarily differentiable. In proving our results, we extend the limiting equation approach to the time-varying delay systems and also develop some new convergence results for functional differential equations.

I. Introduction

Delays are unavoidable in communication, where information has to be transmitted over a physical distance. Unfortunately, very little research has been done to investigate the effect of delays on stability of consensus of multiagent networks. To accurately describe the evolution of networked cooperative systems, it is necessary to include in any mathematical model of the system dynamics some information about the past system states. In this case, the state of the system at a given time involves a portion of trajectories in the space of continuous functions

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defined on an interval of the state space, which leads to (infinite-dimensional) delay dynamical systems [1].

Previously, most of the reported work has either explicitly or implicitly employed the assumption that delays are known and continuously differentiable. Under such an assumption, one can use the delayed state of an agent in its own local control law to match the delays of the states from the neighboring agents [2]–[6], i.e. agent i can use a delayed version of its own state, $x_i(t-\tau_{ij}(t))$. Under that assumption, the control law is

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})), \tag{1}$$

where $\mathcal{N}_i$ denotes the set of all other agents having a communication with agent i. If the delays are constant and uniform, $\tau_{ij}=\tau$ for all ij, then the network dynamics are of the form of time-delayed linear systems with the system matrix being the Laplacian, $\dot{x} = Lx(t-\tau)$, for which various analysis tools for linear systems with delays can be applied [3], [5], [6]. Additionally, the control law in (1) allows one to utilize disagreement dynamics, in which the disagreement $x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})$ is the delayed version of the disagreement $x_j(t)-x_i(t)$. Because of the preceding property, one can study the behavior of the networks using disagreement dynamics or reduced disagreement dynamics in a similar fashion to the case without delays (the reduced disagreement dynamics are asymptotically stable). However, if the delays are unknown, time-varying, and not uniform over the communication links, the assumption that agent i has access to the delayed state $x_i(t-\tau_{ij}(t))$ raises a practical concern. If agent i does not have $x_i(t-\tau_{ij}(t))$ to use in the control protocol (in which case we say that the delays are asymmetric), the control law actually becomes

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)). \tag{2}$$

Because $x_j(t-\tau_{ij}) - x_i(t)$ is no longer the delayed version of the disagreement $x_j(t)-x_i(t)$, the derivatives of the disagreements are not functions of the disagreements only), and hence, the approaches in [3], [5], [6] are not applicable to networks with the protocol (2). Stability of dynamic networks in such a situation has only recently been addressed [7]–[9], most of which are limited to the case of constant time delays. In particular, the authors in [7] have shown that dynamic networks with consensus protocols in the presence of heterogeneous delays are stable for arbitrary constant delays. Another closely related work is [10], where the authors consider

networks with different arrival times for communication and with zero-order hold control laws, which leads to discrete-time dynamic networks formulation without time-delays for the overall closed loop. Left open is the problem of stability and convergence of time-varying consensus dynamic networks in the presence of unknown asymmetric non-uniform time-varying delays, which turns out to be a consequence of the more general results in this paper.

In this paper, we develop a general framework for semistability analysis of nonlinear systems having a continuum of equilibria and time-varying delays in which the delays are unknown and continuous with respect to time, not necessarily continuously differentiable. Here semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. The basic assumption for the main result in this paper involves the idea of *limiting equations* [11] by assuming that the original time-varying delay system asymptotically converges to an autonomous system with constant delays. Using these results, next we present stability analysis of time-varying consensus dynamic networks in the presence of unknown asymmetric non-uniform time-varying delays. The main feature of the proposed framework is that the assumption on continuous differentiability of the time delays is considerably weakened by use of a limiting function assumption, which is more natural and useful in practical systems. The proposed new results can be viewed as a generalization of network consensus with constant time delays in [7].

II. Mathematical Preliminaries

Let $\mathbb{R}^n$ denote the real Euclidean space of n-dimensional column vectors and let $\|x\|$ denote the norm of the vector x in $\mathbb{R}^n$. Let $r \geq 0$ be given and let $\mathcal{C} = \mathcal{C}([-r,0],\mathbb{R}^n)$ denote the space of continuous functions that map the interval $[-r,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $x : [-r,\infty) \to \mathbb{R}^n$ be continuous, then for any $t \geq 0$, $x_t \in \mathcal{C}$ is defined by $x_t(s) = x(t+s)$, $-r \leq s \leq 0$.

Consider nonlinear dynamical systems with time-varying delays given by the form

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau_1(t)),\ldots,x(t-\tau_m(t))), \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and maps closed and bounded sets into bounded sets, $g : \mathcal{C} \to \mathbb{R}^n$ is continuous and maps closed and bounded sets into bounded sets, and $\tau_k : \mathbb{R} \to \mathbb{R}$ is continuous but *not* necessarily differentiable, $k = 1,\ldots,m$. We further assume

that solutions depend continuously on the initial data. Define the equilibrium set of (3) as $\mathcal{E} := \{z \in \mathbb{R}^n : f(z) + g(z, \ldots, z) = 0\}$. Given $\phi \in \mathcal{C}$ and $\tau > 0$, a function $x(\phi)$ is said to be a solution to (3) on $[-r, \tau)$ with initial condition ϕ if $\phi \in \mathcal{C}([-r, \tau), \mathbb{R}^n)$, $x_t \in \mathcal{C}$, $x(t)$ satisfies (3) for $t \in [0, \tau)$ and $x(\phi)(0) = \phi$, where $x(\phi)(\cdot)$ denotes the solution through $(0, \phi)$.

Throughout this paper, we make the following standing assumptions on (3).

Assumption 2.1: $\mathcal{E}$ is a connected set.

Assumption 2.2: For every $k = 1, \ldots, m$, $0 \leq \tau_k(t) \leq h$, where $h > 0$ is a constant.

Recall that a set $\mathcal{E} \subseteq \mathbb{R}^n$ is *connected* if every pair of open sets $\mathcal{U}_i \subseteq \mathbb{R}^n$, $i = 1, 2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \varnothing$, $i = 1, 2$, has a nonempty intersection. Assumption 2.1 implies that (3) has a continuum of equilibria. In other words, the equilibria of (3) are *not* isolated equilibrium points. This situation occurs in many practical problems such as compartmental modeling of biological systems [12], thermodynamic systems [13], multiagent coordinated networks [5], [7], [14], and synchronization of coupled oscillators [8]. Assumption 2.2 implies that time-varying delays for $\mathcal{G}$ are bounded. Using this assumption and the conditions on f and g_k, it follows that (3) possesses a unique solution on $[-h, \infty)$ [1].

Example 2.1: Consider a special case of (3) where $f(x) = Ex$, $g(x, \ldots, x) = \sum_{k=1}^{m} F_k x$, and $E, F_k \in \mathbb{R}^{n \times n}$ are matrices, $k = 1, \ldots, m$. If $E + \sum_{k=1}^{m} F_k$ is singular, then $\mathcal{E}$ is a connected set, i.e., (3) has a continuum of equilibria. A relevant example for this case is the consensus problem with time delays [5], [7], [14] given by the consensus protocol

$$\dot{x}(t) = Ex(t) + \sum_{k=1}^{m} F_k x(t - \tau_k(t)), \tag{4}$$

where $E + \sum_{k=1}^{m} F_k$ is a Laplacian.

Recall that a point $z \in \mathbb{R}^n$ is a positive limit point of a solution $x(t)$ to (3) with $x(s) = \phi(s)$, $-h \leq s \leq 0$, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to +\infty$ and $x(t_n) \to z$ as $n \to +\infty$. The set $\omega(\phi)$ of all such positive limit points is the positive limit set of $x_0 = \phi \in \mathcal{C}$ [1, p. 102]. Motivated by Lemma 2.2 of [15], we have the following result.

Lemma 2.1: Assume that the solutions of (3) are bounded and let $x(\cdot)$ be a solution of (3) with $x_0 = \phi \in \mathcal{C}$. If $z \in \omega(x_0)$ is a Lyapunov stable equilibrium point of (3), then $z = \lim_{t \to \infty} x(t)$ and $\omega(\phi) = \{z\}$.

Proof: Since the solutions of (3) are bounded, it follows from Lemma 1.4 of Chapter 4 of [1] that the positive orbit of (3) is precompact. Hence, it follows from Lemma 1.3 of Chapter 4

of [1] that $\omega(\phi)$ is nonempty. Now the proof of the result is similar to the proof of Lemma 2.2 of [15]. For completeness, we include it here. ■

Definition 2.1: An equilibrium point $x \in \mathcal{E}$ is *semistable* if there exists an open set $\mathcal{U} \subseteq \mathcal{C}$ containing x such that for every initial condition in $\mathcal{U}$, the trajectory of (3) converges, that is, $\lim_{t\to\infty} x(t)$ exists, and every equilibrium point in $\mathcal{U}$ is Lyapunov stable. The system (3) is *semistable* if every equilibrium point in $\mathcal{E}$ is semistable.

III. Main Results

A. General Results for Nonlinear Time Delay Systems

In this section, we use a limiting system approach to study the asymptotic behavior of (3). Specifically, it follows from (3) that

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t-h_1), \ldots, x(t-h_m)) \\ &\quad + g(x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))) \\ &\quad - g(x(t-h_1), \ldots, x(t-h_m)), \end{aligned} \tag{5}$$

where $0 \leq h_k \leq h$, $k = 1, \ldots, m$, are some constants that are *not* necessarily known. Next, define $\mathcal{X}(t) := g(x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))) - g(x(t-h_1), \ldots, x(t-h_m))$. Then we have

$$\dot{x}(t) = f(x(t)) + g(x(t-h_1), \ldots, x(t-h_m)) + \mathcal{X}(t). \tag{6}$$

Note that if $x(t) \equiv \alpha \in \mathcal{E}$, then $\mathcal{X}(t) = 0$.

Definition 3.1: If there exists $\mathcal{D} \subseteq \mathcal{C}$ such that for every initial condition $x_0 = \phi \in \mathcal{D}$,

$$\lim_{t\to\infty} \mathcal{X}(t) = 0, \tag{7}$$

then the system

$$\dot{z}(t) = f(z(t)) + g(z(t-h_1), \ldots, z(t-h_m)) \tag{8}$$

with the initial condition $z_0 \in \mathcal{D}$ is called a *limiting system* of (6) with respect to $\mathcal{D}$. If, in addition, $\mathcal{D} = \mathcal{C}$, then we simply say that (8) is a limiting system of (6).

Remark 3.1: The idea of the limiting equation approach was originally from [11] and has been extended to various finite-dimensional dynamical systems by changing the definition of limiting functions [16], [17]. Our definition extends this approach to infinite-dimensional dynamical systems and gives a new definition of limiting systems for time-delay systems.

Note that the limiting system (8) has the same equilibrium set as (6) or, equivalently, (3). Based on the notion of limiting systems, we have the following convergence result.

Lemma 3.1: Consider the nonlinear system (6). Assume the trajectories of (6) are bounded. Furthermore, assume (8) is a limiting system of (6). Then $\omega(\phi)$ is invariant with respect to (8) for every initial condition $x_0 = \phi \in \mathcal{C}$.

Proof: Since the solutions of (6) are bounded, it follows that the set $\omega(\phi)$ is nonempty. Let $z \in \omega(\phi)$ and hence, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ and $x(t_n) \to z$ as $n \to \infty$. For each $n = 1, 2, \ldots$, define the continuous functions $x_n : [0, \infty) \to \mathbb{R}^n$ and $f_n : [0, \infty) \to [0, \infty)$ as

$$x_n(t) := x(t + t_n), \quad t \geq 0, \tag{9}$$

$$f_n(t) := \|\mathcal{X}_n(t)\| + \frac{1}{n}, \quad t \geq 0, \tag{10}$$

where

$$\mathcal{X}_n(t) := g(x_n(t - \tau_1(t)), \ldots, x_n(t - \tau_m(t))) - g(x_n(t - h_1), \ldots, x_n(t - h_m)). \tag{11}$$

We claim that f_n is measurable. To this end, note that for each n, continuity of the map $t \mapsto R(\mathfrak{D}x_n(t))$, where $R(\mathfrak{D}x(t)) := g(x(t - h_1), \ldots, x(t - h_m))$, together with compactness of its values ensures that for each $z \in \mathbb{R}^n$, the map $t \mapsto \|z - R(\mathfrak{D}x_n(t))\|$ is measurable. Next, since the function $t \mapsto g(x_n(t - \tau_1(t)), \ldots, x_n(t - \tau_m(t)))$ is measurable, it follows that this function is the pointwise limit of a sequence of simple functions (i.e., measurable functions with finite images). Now, it follows that f_n is the pointwise limit of a sequence of measurable functions and hence, is measurable.

Note that for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $n > N$, $0 < f_n(t) < \varepsilon$ for all $t \geq 0$. Observe that $x_n(0) \to z$ as $n \to \infty$. Then it follows that for all $n > N$, we have

$$\begin{aligned} \dot{x}_n(t) &\in \mathcal{B}_{f_n(t)}(f(x_n(t)) + R(\mathfrak{D}x_n(t))) \\ &\subset \mathcal{B}_{\varepsilon}(f(x_n(t)) + R(\mathfrak{D}x_n(t))) \text{ a.e. } t \geq 0, \end{aligned} \tag{12}$$

which represents a set-valued differential equation called *differential inclusion* in the literature [18], [19], where $\mathcal{B}_r(s)$ denotes the open ball centered at s with radius r and "a.e." denotes almost everywhere in the sense of Lebesgue measure.

Let I_r denote the interval $[r, r+1]$. Using (12) and by the time-delay version of Theorem 3.1.7 of [18] or Lemma 4.5 of [20], the sequence of restricted functions $\{x_n(t)\}$, $t \in I_0$, has a subsequence $\{x_{\sigma_1(n)}(t)\}$, $t \in I_0$, converging uniformly as $n \to \infty$ to an absolutely continuous function $x_1^* : I_0 \to \mathbb{R}^n$, satisfying (8) almost everywhere and with $x_1^*(0) = z$. By repeating the same argument, the sequence $\{x_{\sigma_1(n)}(t)\}$, $t \in I_1$, contains a subsequence $\{x_{\sigma_2(n)}(t)\}$, $t \in I_1$, converging uniformly as $n \to \infty$ to an absolutely continuous function $x_2^* : I_1 \to \mathbb{R}^n$, satisfying (8) almost everywhere and with $x_2^*(1) = x_1^*(1)$. By induction, there exists a nested sequence $\{x_{\sigma_\ell(n)}\}_{\ell=1}^{\infty}$ of subsequences of $\{x_n\}$ such that for each $\ell = 1, 2, \ldots$, $\{x_{\sigma_\ell(n)}\}$ converges uniformly on $I_{\ell-1} = [\ell - 1, \ell]$ to an absolutely continuous function $x_\ell^* : I_{\ell-1} \to \mathbb{R}^n$ that satisfies (8) almost everywhere and with $x_{\ell+1}^*(\ell) = x_\ell^*(\ell)$ and $x_1^*(0) = z$.

Define $x^* : [0, \infty) \to \mathbb{R}^n$ as $x^*(t) = x_\ell^*(t)$ for $t \in I_{\ell-1}$. Then $x^*(\cdot)$ satisfies (8) and $x^*(0) = z$. Using the Cantor diagonal argument (see, e.g., [21, p.210]), it follows that there exists a subsequence $\{x_{\sigma_n(n)}\}$ of $\{x_n\}$ converging uniformly on every interval $[0, \ell]$ to x^* for all ℓ. Since the sequence $\{t_{\sigma_n(n)}\}$ is such that $t_{\sigma_n(n)} \to \infty$ and $x(t + t_{\sigma_n(n)}) = x_{\sigma_n(n)}(t) \to x^*(t)$ as $n \to \infty$, it follows that $x^*(t) \in \omega(x_0)$ for all $t \geq 0$, which implies that $\omega(x_0)$ is invariant with respect to (8). ■

Lemma 3.2: Consider (8). If the trajectories of (8) converge, that is, $\lim_{t\to\infty} z_t(\phi)$ exists for every $\phi \in \mathcal{C}$, then the function $\Omega : \mathcal{C} \to \mathcal{C}$ defined by $\Omega(\phi) = \lim_{t\to\infty} z_t(\phi)$, $\phi \in \mathcal{C}$, is an equilibrium point for (8).

Proof: It follows from continuity of the solutions to (8) that for every $s \geq 0$, $z_s(\Omega(\phi)) = \lim_{t\to\infty} z_{t+s}(\phi) = \Omega(\phi)$. Thus, $\Omega(\phi)$ is an equilibrium point for (8) and for all $\phi \in \mathcal{C}$. ■

Now we have the main result for this paper.

Theorem 3.1: Consider the nonlinear system (6). Assume (6) is Lyapunov stable. Furthermore, assume (8) is a limiting system of (6) and (8) is semistable. Then (6) is semistable.

Proof: Since by assumption, (6) is Lyapunov stable, it follows that the trajectories of (6) are bounded. Then by Lemma 3.1, $\omega(\phi)$ is invariant with respect to (8). Next, since by semistability, the trajectories of (8) converge, it follows that $\omega(\phi)$ contains positive limit points of (8), and hence, $\omega(\phi)$ contains the positive limit set of (8). Furthermore, it follows from Lemma 3.2 that the positive limit set of (8) contains an equilibrium point of (8). This equilibrium point is also an equilibrium point of (6) and by assumption, it is Lyapunov stable. Hence, $\omega(\phi)$ contains a Lyapunov stable equilibrium point for (6). Now it follows from Lemma 2.1 that the trajectory

of (6) converges to this Lyapunov stable equilibrium point, which implies convergence of the trajectories of (6). By definition, (6) is semistable. ■

Remark 3.2: To discuss semistability of (6) using Theorem 3.1, one has to know the information on Lyapunov stability of (6). Note that here we only assume $\tau_k(t)$ is *continuous* for every $k = 1, \ldots, m$. Hence, it is very difficult to use the Lyapunov-Krasovskii functional approach [1], [22] to prove the Lyapunov stability of (6) since it requires the first-order derivative of $\tau_k(t)$. In this case, the Lyapunov stability of (6) may be verified using Razumikhin theorems via Lyapunov-Razumikhin functions [1], [23], [24].

Example 3.1: Consider the scalar time-delay system given by

$$\dot{x}(t) = -x(t) + x(t - \tau(t)), \tag{13}$$

where $x(t) \in \mathbb{R}$, $\tau(\cdot)$ is continuous, and $0 \leq \tau(t) \leq h$ for all $t \in \mathbb{R}$. Consider the Lyapunov-Razumikhin function given by $V(x) = (x - \alpha)^2/2$, where α is an arbitrary constant. Then it follows from Theorem 4.1 of Chapter 5 of [1] that (13) is uniformly Lyapunov stable. See [1, p. 154] for a detailed proof.

Remark 3.3: Suppose the trajectories of (6) are bounded. If $\|f(x)\| \leq \beta(\|x\|)$, $\beta(\cdot)$ is a class $\mathcal{K}$ function, $g(x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) = \sum_{k=1}^{m} g_k(x(t - \tau_k(t)))$, $g_k(\cdot)$ is globally Lipschitz continuous, $k = 1, \ldots, m$, and $\lim_{t\to\infty} \tau_k(t) = h_k$ for every $k = 1, \ldots, m$, then (8) is a limiting system of (6). To see this, suppose $\|x(t)\| \leq M$. Then from (6), $\|\dot{x}(t)\| \leq \beta(M) + \sum_{k=1}^{m}(L_k M + \|g_k(0)\|) := K$, where L_k is the Lipschitz constant, $k = 1, \ldots, m$. Because $x(t - \tau_k(t)) - x(t - h_k) = \int_{t-h_k}^{t-\tau_k(t)} \dot{x}(s)ds$, it follows that $\|x(t - \tau_k(t)) - x(t - h_k)\| \leq K|\tau_k(t) - h_k|$. Hence, $\|g_k(x(t - \tau_k(t))) - g_k(x(t - h_k))\| \leq L_k\|x(t - \tau_k(t)) - x(t - h_k)\| \leq KL_k|\tau_k(t) - h_k|$. Thus, if $\lim_{t\to\infty} \tau_k(t) = h_k$, then $\lim_{t\to\infty} \mathcal{X}(t) = 0$. By definition, (18) is a limiting system of (4).

Example 3.2: Consider the time-delay system given by (13) where $\tau(t) = h|\sin(\pi/2 + \pi/(1 + |t|))|$, $t \in \mathbb{R}$. Clearly $\tau(\cdot)$ is continuous but not differentiable for all $t \in \mathbb{R}$. We claim that

$$\dot{z}(t) = -z(t) + z(t - h) \tag{14}$$

is a limiting system of (13). To see this, note that $\lim_{t\to\infty} \tau(t) = h$. Now it follows from Remark 3.3 that (14) is a limiting system of (13).

Next, motivated by [15], we present a Lyapunov-type result for semistability of nonlinear systems with constant time delays using Lyapunov-Krasovskii functionals. This result will help us determine the semistability of (8) which is required by Theorem 3.1.

Theorem 3.2: Consider the dynamical system (8). Assume the trajectories of (8) are bounded and there exists a continuous functional $V : \mathcal{C} \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathcal{C}$ and $\dot{V}(\phi) \leq 0$ for all $\phi \in \mathcal{C}$. If every point in the largest invariant set $\mathcal{M}$ of $\dot{V}^{-1}(0)$ is a Lyapunov stable equilibrium point of (8), then (8) is semistable.

Proof: Since every solution is bounded, it follows from the hypotheses on V that, for every $\phi \in \mathcal{C}$, the positive limit set of (8) denoted by $\varpi(\phi)$ is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\dot{V}^{-1}(0)$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium of (8), it follows from Lemma 2.1 that $\varpi(\phi)$ contains a single point for every $\phi \in \mathcal{C}$ and the trajectories of (8) converge. Since $\Omega(\phi)$ is Lyapunov stable for every $\phi \in \mathcal{C}$, semistability follows. ■

Example 3.3: Consider the time-delay system given by (14). Let $V(z_t) = z^2(t) + \int_{t-h}^{t} z^2(s)\mathrm{d}s$. Then the derivative of $V(\cdot)$ along the trajectories of (14) is given by $\dot{V}(z_t) = 2z(t)\dot{z}(t) + z^2(t) - z^2(t-h) = -(-z(t) + z(t-h))^2 \leq 0$, $t \geq 0$. Note that $\dot{V}^{-1}(0) = \{\phi(\cdot) \in \mathcal{C} : -\phi(0) + \phi(-h) = 0\}$. Furthermore, the largest invariant set contained in $\dot{V}^{-1}(0)$ is given by $\mathcal{M} = \{\phi(\cdot) \in \mathcal{C} : \phi(\theta) = \alpha \in \mathbb{R}, \theta \in [-h, 0]\}$. Now, it follows from Example 3.1 and Theorem 3.2 that (14) is semistable, and hence, by Example 3.1 and Theorem 3.1, (13) with $\tau(t) = h|\sin(\pi/2 + \pi/(1+|t|))|$ is semistable.

As an alternative to Theorem 3.2, we present a Lyapunov-Razumikhin function approach to semistability analysis of nonlinear systems with constant time delays. Motivated by [25], this result gives a different method to prove semistability of (8) other than Theorem 3.2, which is useful for many cases in that constructing a Lyapunov-Krasovskii functional for (8) may not be an easy task in these cases.

Theorem 3.3: Consider the dynamical system (8). Assume the trajectories of (8) are bounded and there exists a continuous function $V : \mathcal{C} \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathcal{C}$ and $\dot{V}(\phi) \leq 0$ for all $\phi \in \mathcal{C}$ such that $V(\phi(0)) = \max_{-h \leq s \leq 0} V(\phi(s))$. If every point in the largest invariant set $\mathcal{M}$ of $\mathcal{R} := \{\phi \in \mathcal{C} : \max_{s\in[-h,0]} V(z_t(\phi)(s)) = \max_{s\in[-h,0]} V(\phi(s)), \forall t \geq 0\}$ is a Lyapunov stable equilibrium point of (8), then (8) is semistable.

Proof: Let $\phi \in \mathcal{C}$ be such that $z_t(\phi)$ is bounded on $[-h, \infty)$. Then $\varpi(\phi)$ is nonempty. Using a standard Razumikhin-type argument (see the proof of Theorem 4.1 in [1, p. 152]) and the

assumptions on V, it follows that the function $\max_{-h\leq s\leq 0} V(z_t(\phi)(s))$ is a nonincreasing function of t on $[0,\infty)$. Since V is bounded from below along this solution, $\lim_{t\to\infty}\{\max_{-h\leq s\leq 0} V(z_t(\phi)(s))\}$ exists. Hence, $\varpi(\phi)\subseteq\mathcal{M}\subseteq\mathcal{R}$ and $z_t(\phi)\to\mathcal{M}$ as $t\to\infty$. Finally, since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point of (8), it follows from Lemma 2.1 that the trajectories of (8) converge. Thus, by definition, (8) is semistable. ■

Example 3.4: Consider the scalar time-delay system given by

$$\dot{x}(t) = -mx(t) + \sum_{k=1}^{m} x(t-\tau_k(t)), \tag{15}$$

where $x(t)\in\mathbb{R}$ and $\tau_k(t) = (hk/m)|\sin(\pi/2+\pi/(1+|t|))|$ for every $k=1,\ldots,m$, $t\in\mathbb{R}$. Using the Lyapunov-Razumikhin function $V(x)=(x-\alpha)^2/2$ and similar arguments as in [1, p. 154], it follows that (15) is uniformly Lyapunov stable. Next, note that

$$\dot{z}(t) = -mz(t) + \sum_{k=1}^{m} z(t-hk/m) \tag{16}$$

is a limiting system of (15). We show that (16) is semistable. To see this, note that for $V(z)=(z-\alpha)^2/2$, $\alpha\in\mathbb{R}$, we have

$$\begin{aligned}\dot{V}(z(t)) &= -mz^2(t) + \sum_{k=1}^{m} z(t)z(t-hk/m)\\ &\leq -mz^2(t) + \sum_{k=1}^{m} |z(t)||z(t-hk/m)|\\ &\leq -mz^2(t) + \sum_{k=1}^{m} z^2(t) = 0\end{aligned} \tag{17}$$

if $|z(t+\theta)|\leq|z(t)|$ for $\theta\in[-h,0]$. Hence, it follows from Theorem 4.1 of [1, p. 152] that (16) is Lyapunov stable. Next, we want to compute $\mathcal{R}$ and $\mathcal{M}$. First of all, note that $\mathcal{M}$ is nonempty since $0\in\mathcal{M}$. Let $\phi\in\mathcal{R}$, that is, let $\phi\in\mathcal{C}$ be such that $\max_{-h\leq\theta\leq 0}|z_t(\phi)(\theta)| = \max_{-h\leq\theta\leq 0}|\phi(\theta)|$ for all $t\geq 0$. For $\phi\in\mathcal{R}$ satisfying $|\phi(0)|\geq|\phi(s)|$, $s\in[-h,0]$, there exists $t^*>0$ for which $\dot{V}(z_{t^*}(\phi))=0$ as V attains a relative maximum for such t^*. For such a t^*, $-mz^2(t^*)+\sum_{k=1}^{m} z(t^*)z(t^*-hk/m)=0$, and hence, $z(t^*)=0$ or $-mz(t^*)+\sum_{k=1}^{m} z(t^*-hk/m)=0$. Consider the case where $\phi(0)\geq 0$. Then it follows that $z(t^*+\theta)\leq z(t^*)$ for all $\theta\in[-h,0]$. Therefore, $z(t^*)=0$ or $z(t^*)=z(t^*-hk/m)$ for every $k=1,\ldots,m$. Due to uniqueness of solutions, we have $z_t(\phi)=0$ or $z_t(\phi)=z_t(\phi)(-kh/m)$ for every $k=1,\ldots,m$ and all $t\geq t^*$. By Example 3.3, it follows that $\mathcal{M}=\{\phi(\cdot)\in\mathcal{C}:\phi(\theta)=\alpha\in\mathbb{R},\theta\in[-h,0]\}$. Now, it follows

from Theorem 3.3 that (16) is semistable. Finally, it follows from Theorem 3.1 that (15) is semistable.

B. Specialization to the Consensus Problem

Lemma 3.3: Consider the dynamical system (4). Assume the trajectories of (4) are bounded. If $\lim_{t\to\infty}\tau_k(t)=h_k$ for every $k=1,\ldots,m$, then

$$\dot{z}(t)=Ez(t)+\sum_{k=1}^{m}F_kz(t-h_k) \tag{18}$$

is a limiting system of (4).

Proof: The proof is essentially given by Remark 3.3. For completeness, we include it here. Suppose $\|x(t)\|\leq M$. Then from (4), $\|\dot{x}(t)\|\leq\|E\|M+\sum_{k=1}^{m}\|F_k\|M:=K$. Because $x(t-\tau_k(t))-x(t-h_k)=\int_{t-h_k}^{t-\tau_k(t)}\dot{x}(s)ds$, it follows that $\|x(t-\tau_k(t))-x(t-h_k)\|\leq K|\tau_k(t)-h_k|$. Thus, if $\lim_{t\to\infty}\tau_k(t)=h_k$, then $\lim_{t\to\infty}\mathcal{X}(t)=0$. By definition, (18) is a limiting system of (4). ■

Next, we present a Lyapunov stability result for (4). Define $F:=\sum_{k=1}^{m}F_k$. For a matrix $A\in\mathbb{R}^{m\times n}$, we use $A_{(i,j)}$ to denote the (i,j)th element of A.

Lemma 3.4: Consider the dynamical system (4) having the following structure: all the elements in F_k are nonnegative, $k=1,\ldots,m$,

$$E_{(i,j)} = \begin{cases} -\sum_{k=1}^{n}a_{ki}, & i=j,\\ 0, & i\neq j,\end{cases} \tag{19}$$

$$F_{(i,j)} = \begin{cases} 0, & i=j,\\ a_{ij}, & i\neq j,\end{cases} \tag{20}$$

$a_{ij}\geq 0$, $i,j=1,\ldots,n$. Then (4) is Lyapunov stable.

Proof: Note that under the assumptions in Lemma 3.4, (4) can be rewritten as

$$\begin{aligned}\dot{z}_i(t) &= -\sum_{k=1}^{m}\sum_{j=1}^{n}F_{k(i,j)}z_i(t)+\sum_{k=1}^{m}\sum_{j=1}^{n}F_{k(i,j)}z_j(t-\tau_k(t))\\ &= \sum_{k=1}^{m}\sum_{j=1}^{n}F_{k(i,j)}(z_j(t-\tau_k(t))-z_i(t)),\end{aligned} \tag{21}$$

where $F_{k(i,j)}\geq 0$ for all $k=1,\ldots,m$ and $i,j=1,\ldots,n$.

Consider the Lyapunov-Razumikhin function given by

$$V(\phi) = \frac{1}{2}\max_{1\le i\le n}\{(\phi_i-\alpha)^2\}, \tag{22}$$

where $\alpha \in \mathbb{R}$. To use Razumikhin theorems ([1, p. 151]) showing Lyapunov stability, we focus on $V(\phi(0)) = \max_{-h\le s\le 0} V(\phi(s))$, that is, for the cases in which

$$(\phi_I(0)-\alpha)^2 \ge (\phi_j(s)-\alpha)^2, \quad s\in[-h,0], \tag{23}$$

where I is the index for which $|\phi_I-\alpha| = \max_{1\le i\le n}|\phi_i-\alpha|$.

The derivative of V along the trajectories of (4) is given by $\dot{V}(\phi) = (\phi_I(0)-\alpha)\dot{\phi}_I(0)$. First suppose $\phi_I(0)-\alpha = c > 0$. Then it follows from (23) that $-c \le \phi_j(s)-\alpha \le c$ for all $j = 1,\dots,n$ and $s\in[-h,0]$. This implies that $\phi_j(s)-\phi_I(0)\le 0$ for all $j=1,\dots,n$ and $s\in[-h,0]$. Hence, by (21), it follows that $\dot{\phi}_I(0)\le 0$. Similarly, one can show that if $\phi_I(0)-\alpha\le 0$, then $\dot{\phi}_I(0)\ge 0$. In summary, $\dot{V}(\phi)\le 0$. Now it follows from Theorem 4.1 of [1, p. 152] that (4) is Lyapunov stable. ■

The following corollary regarding semistability of time-varying delay network consensus protocols given by (4) follows directly from Lemmas 3.3 and 3.4, Theorem 3.1, and a result from [7]. To state this result, define $\mathbf{1} = [1,\dots,1]^{\mathrm{T}}\in\mathbb{R}^n$.

Corollary 3.1: Consider the dynamical system (4) having the structure given by (19) and (20). Assume $\lim_{t\to\infty}\tau_k(t) = h_k$ for every $k=1,\dots,m$. Furthermore, assume that $(E+F)^{\mathrm{T}}\mathbf{1} = (E+F)\mathbf{1} = 0$ and rank$(E+F) = n-1$. Then for every $\alpha\in\mathbb{R}$, $\alpha\mathbf{1}$ is a semistable equilibrium point of (4). Furthermore, $x(t)\to\alpha^*\mathbf{1}$ as $t\to\infty$, where

$$\alpha^* = \frac{\mathbf{1}^{\mathrm{T}}\phi(0)+\sum_{k=1}^m\int_{-h_k}^0\mathbf{1}^{\mathrm{T}}F_k\phi(\theta)\mathrm{d}\theta}{n+\sum_{k=1}^m h_k\mathbf{1}^{\mathrm{T}}F_k\mathbf{1}}. \tag{24}$$

Proof: First, it follows from Lemma 3.3 that

$$\dot{x}(t) = Ex(t)+\sum_{k=1}^m F_k x(t-h_k) \tag{25}$$

is a limiting system of (4). Next, it follows from Theorem 3.1 of [7] that (25) is semistable. Now, it follows from Lemma 3.4 and Theorem 3.1 that (4) is semistable. The expression (24) for α^* follows from Theorem 3.1 of [7]. ■

Example 3.5: Consider the time-delay system given by

$$\dot{x}_1(t) = -x_1(t)+x_2(t-\tau_1(t)), \tag{26}$$

$$\dot{x}_2(t) = -x_2(t)+x_1(t-\tau_2(t)), \tag{27}$$

where $\tau_1(t) = h_1|t\sin(1/t)|$ for $t \neq 0$ and $\tau_1(t) = h_1$ for $t = 0$, and $\tau_2(t) = h_2(1 - e^{-|t|})$. Clearly in this case,

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now it follows from Corollary 3.1 that the time-delay system given by (26) and (27) is semistable.

Next, we generalize Corollary 3.1 to the nonlinear system given by

$$\dot{x}(t) = f(x(t)) + \sum_{k=1}^{m} g_k(x(t-\tau_k(t))), \tag{28}$$

where $f = [f_1, \ldots, f_q]^{\mathrm{T}}$. Using some result from [7], we have the following stability result for the nonlinear network consensus with time-varying delays given by the form of (28). Recall that for a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$, the *Drazin inverse* $\Lambda^{\mathrm{D}} \in \mathbb{R}^{n\times n}$ is given by $\Lambda^{\mathrm{D}}_{(i,i)} = 0$ if $\Lambda_{(i,i)} = 0$ and $\Lambda^{\mathrm{D}}_{(i,i)} = 1/\Lambda_{(i,i)}$ if $\Lambda_{(i,i)} \neq 0$, $i = 1, \ldots, n$ [26, p. 227].

Corollary 3.2: Consider the dynamical system (28) where $f(0) = 0$, $g_k(0) = 0$, $k = 1, \ldots, m$, and $f_i(\cdot)$ is strictly decreasing for $f_i \not\equiv 0$, $i = 1, \ldots, n$. Assume (28) is Lyapunov stable and $\lim_{t\to\infty} \tau_k(t) = h_k$ for every $k = 1, \ldots, m$. Next, assume that $\mathbf{1}^{\mathrm{T}}(f(x) + \sum_{k=1}^{m} g_k(x)) = 0$ for all $x \in \mathbb{R}^n$ and $f(x) + \sum_{k=1}^{m} g_k(x) = 0$ if and only if $x = c\mathbf{1}$ for some $c \in \mathbb{R}$. Furthermore, assume that there exist nonnegative diagonal matrices $P_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, m$, such that $P := \sum_{k=1}^{m} P_k > 0$, $P_k^{\mathrm{D}} P_k g_k(x) = g_k(x)$ for every $x \in \mathbb{R}^n$ and $k = 1, \ldots, m$, and

$$\sum_{k=1}^{m} g_k^{\mathrm{T}}(x) P_k g_k(x) \quad \leq \quad f^{\mathrm{T}}(x) P f(x), \quad x \in \mathbb{R}^n, \tag{29}$$

$$\sum_{k=1}^{m} f^{\mathrm{T}}(x) P P_k^{\mathrm{D}} P f(x) \quad \leq \quad f^{\mathrm{T}}(x) P f(x), \quad x \in \mathbb{R}^n. \tag{30}$$

Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{1}$ is a semistable equilibrium point of (28). Furthermore, $x(t) \to \alpha^*\mathbf{1}$ as $t \to \infty$, where α^* satisfies

$$n\alpha^* + \sum_{k=1}^{m} h_k \mathbf{1}^{\mathrm{T}} g_k(\alpha^*\mathbf{1}) = \mathbf{1}^{\mathrm{T}}\phi(0) + \sum_{k=1}^{m} \int_{-h_k}^{0} \mathbf{1}^{\mathrm{T}} g_k(\phi(\theta))\mathrm{d}\theta. \tag{31}$$

Proof: First, it follows from Lemma 3.3 that

$$\dot{x}(t) = f(x(t)) + \sum_{k=1}^{m} g_k(x(t-h_k)) \tag{32}$$

is a limiting system of (28). Next, it follows from Theorem 4.1 of [7] that (32) is semistable. Now, it follows from Theorem 3.1 that (28) is semistable. The equation (31) follows from Theorem 4.1 of [7]. ■

Example 3.6: Consider the time-delay system given by

$$\dot{x}_1(t) = -x_1^3(t) + x_2^3(t-\tau_1(t)), \tag{33}$$

$$\dot{x}_2(t) = -x_2^3(t) + x_1^3(t-\tau_2(t)), \tag{34}$$

where $\tau_1(t) = h_1 - h_1 e^{-|t|}\sin t$ and $\tau_2(t) = h_2 - h_2 \sin(1/(1+|t|))$. In this case,

$$f(x) = \begin{bmatrix} -x_1^3 \\ -x_2^3 \end{bmatrix},\ g_1(x) = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix},\ g_2(x) = \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now it follows from Corollary 3.2 that the time-delay system given by (33) and (34) is semistable.

IV. Conclusion

A new framework concerning semistability of nonlinear systems having a continuum of equilibria and time-varying delays is presented and its applications to stability analysis of multiagent dynamic networks with consensus protocol in the presence of unknown heterogeneous time-varying delays are discussed in this paper. Those time delays are not necessarily differentiable and known. We provided conditions, in terms of the limiting system, to guarantee semistability of nonlinear systems with multiple time-varying delays and applied those stability results to show that multiagent dynamic networks can still achieve consensus in the presence of heterogeneous delays, provided that the delays converge to a limit asymptotically.

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